Name: ____________________ GTID: ____________________

- Fill out your name and Georgia Tech ID number.
- This quiz contains 4 pages. Please make sure no page is missing.
- The grading will be done on the scanned images of your test. Please write clearly and legibly.
- Answer the questions in the spaces provided. We will scan the front sides only by default. If you run out of room for an answer, continue on the back of the page and notify the TA when handing in.
- Please write detailed solutions including all steps and computations; use explanations written in words, if necessary.
- The duration of the quiz is 40 minutes.

Good luck!

1. (30 points) Consider an RLC-circuit and recall that the differential equation for q, the charge on the capacitor, is

$$Lq'' + Rq' + \frac{q}{C} = 0$$

for constants $L, R, C \geq 0$ depending on the electrical components. Let $L = 3$ and $C = \frac{1}{3}$.

(a) Let $R = 3$ in this part. Find the charge on the capacitor as a function of time.

(b) What is the threshold for R such that the solution does not admit any oscillatory behavior anymore, i.e. does not contain either sin or cos?[1]

Solution:

(a) The differential equation is $3q'' + 3q' + 3q = 0$ and the corresponding characteristic equation is $3\lambda^2 + 3\lambda + 3 = 0$. The roots of this polynomial can be calculated as

$$\lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 36}}{6}.$$

With $R = 3$, we get $\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Thus, the solution to the differential equation is

$$q(t) = c_1 e^{-0.5t} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_1 e^{-0.5t} \sin\left(\frac{\sqrt{3}}{2}t\right),$$

for constants $c_1, c_2 \in \mathbb{R}$.

(b) The system admits oscillatory behavior if and only if the eigenvalues have non-zero imaginary part. Checking the above equation for the eigenvalues, we see that this is the case if and only if $R^2 - 36 < 0$ which is equivalent to $R < 6$. Therefore, $R = 6$ is the threshold.

[1] A system with this dampening is called "critically damped".

2. (35 points) Solve the ODE

$$y'' + 3y' + 2y = 20 \sin 2t$$

with the initial value problem $y(0) = y'(0) = 0$.

Note that the homogeneous ODE $z'' + 3z' + 2z = 0$ has the solution

$$z(t) = c_1 e^{-t} + c_2 e^{-2t};$$

you do not need to do that part of the work (to save you some time.)

Solution:

We need to find a particular solution of the original equation. We use the method of undetermined coefficients. Search for the solution in the form

$$y_c(t) = A \sin 2t + B \cos 2t.$$

We have

$$y_c'(t) = 2A \cos 2t - 2B \sin 2t,$$

and

$$y_c''(t) = -4A \sin 2t - 4B \cos 2t.$$

Hence

$$y_c'' + 3y_c' + 2y_c = (-4A - 6B + 2A) \sin 2t + (-4B + 6A + 2B) \cos 2t,$$

and in order for it to be equal to $20 \sin 2t$ we need $-2A - 6B = 20$ and $-2B + 6A = 0$. Thus $A = -1$ and $B = -3$. In summary, the general solution of our ODE is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} - \sin 2t - 3 \cos 2t.$$

Finally,

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t} - 2 \cos 2t + 6 \sin 2t.$$

Thus $0 = y(0) = c_1 + c_2 - 3$ and $0 = y'(0) = -c_1 - 2c_2 - 2$, which leads to $c_1 = 8$ and $c_2 = -5$. Hence the answer is

$$y(t) = 8e^{-t} - 5e^{-2t} - \sin 2t - 3 \cos 2t.$$

3. (35 points) Find the general solution to the ODE

$$y'' - 4y' + 4y = \frac{e^{2t}}{1+t^2}.$$

Solution: We first determine a fundamental set of solutions to the corresponding homogeneous ODE

$$y'' - 4y' + 4y = 0.$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0,$$

which has repeated root $\lambda = 2$. Therefore $y_1(t) = e^{2t}, y_2(t) = te^{2t}$ form a fundamental set of solutions. Now we compute a particular solution to the original ODE. The wronskian for the fundental set of solutions is

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix} = e^{4t}.$$

Using the variation of parameters method, a particular solution for the ODE is

$$\begin{aligned} y_p &= -y_1(t) \int \frac{y_2(t)g(t)}{W[y_1, y_2]}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{W[y_1, y_2]}\, dt \\ &= -e^{2t} \int \frac{te^{2t}e^{2t}}{e^{4t}(1+t^2)}\, dt + te^{2t} \int \frac{e^{2t}e^{2t}}{e^{4t}(1+t^2)}\, dt \\ &= -e^{2t} \int \frac{t}{1+t^2}\, dt + te^{2t} \int \frac{1}{1+t^2}\, dt \\ &= \frac{1}{2} e^{-2t} \log(1+t^2) + te^{2t} \tan^{-1}(t). \end{aligned}$$

Therefore the general solution to the ODE is

$$y = c_1 y_1(t) + c_2 y_2(t) + y_p = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{2} e^{-2t} \log(1+t^2) + te^{2t} \tan^{-1}(t).$$